

WHEN IS THE SUM OF COMPLEMENTED SUBSPACES COMPLEMENTED?

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ABSTRACT. We provide a sufficient condition for the sum of a finite number of complemented subspaces of a Banach space X to be complemented. Under this condition a formula for a projection onto the sum is given. We also show that the condition is sharp (in a certain sense). Special attention is paid to the case when X is a Hilbert space; in this case we get more precise results.

1. INTRODUCTION

1.1. Let X be a (complex or real) Banach space and X_1, \dots, X_n be complemented subspaces of X . Define the sum of X_1, \dots, X_n in the natural way, namely,

$$X_1 + \dots + X_n := \{x_1 + \dots + x_n \mid x_1 \in X_1, \dots, x_n \in X_n\}.$$

The natural question arises:

Question 1:

Is $X_1 + \dots + X_n$ complemented in X ?

Note that Question 1 makes sense — the sum of two complemented subspaces may be uncomplemented and even nonclosed. A simple example: let X be a Hilbert space, then a subspace is complemented if and only if it is closed, and there are well-known simple examples of two closed subspaces with nonclosed sum. Note that even if the sum of two complemented subspaces is closed, it can be uncomplemented. An example: let Y be a closed uncomplemented subspace of a Banach space Z ; take $X = Y \times Z$,

$$X_1 = \{(y, 0) \mid y \in Y\}, \quad X_2 = \{(y, y) \mid y \in Y\}.$$

It is easily seen that X_1 and X_2 are complemented in X but the sum $X_1 + X_2 = Y \times Y$ is not.

1.2. If Question 1 has positive answer, then the next natural question arises:

Question 2:

Suppose that we know some (continuous linear) projections P_1, \dots, P_n onto X_1, \dots, X_n , respectively. Is there a formula for a projection onto $X_1 + \dots + X_n$ (in terms of P_1, \dots, P_n) (of course, under certain conditions)?

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1.3. Since a complemented subspace is necessarily closed, Question 1 is closely related to the following

Question 3:

Is $X_1 + \dots + X_n$ closed in X ?

It is worth mentioning that if X is a Hilbert space, then Question 1 coincides with Question 3.

Systems of subspaces X_1, \dots, X_n for which Question 3 is very important arise in various branches of mathematics, for example, in

- (1) theoretical tomography and theory of ridge functions (plane waves). Here the problem on the closedness of the sum of spaces of functions, which are constant on certain sets, naturally arises. See, e.g., [29, 21, 7, 18, 22];
- (2) theory of wavelets and multiresolution analysis. Here the problem on the closedness of the sum of shift-invariant subspaces of $L^2(\mathbb{R}^d)$ is studied. See, e.g., [17] and references therein;
- (3) statistics. See, e.g., [6], where the closedness of the sum of two marginal subspaces is important for constructing an efficient estimation of linear functionals of a probability measure with known marginal distributions;
- (4) projection algorithms for solving convex feasibility problems (problems of finding a point in the nonempty intersection of n closed convex sets), and, in particular, methods of alternating projections. See, e.g., [3, Theorem 5.19], [2, Theorem 4.1], [25], [23] and the bibliography therein;
- (5) a problem of finding an element of a Hilbert space with prescribed best approximations from a finite number of subspaces. This problem is a common problem in applied mathematics, it arises in harmonic analysis, optics, and signal theory. See, e.g., [8] and references therein;
- (6) theory of Banach algebras. See, e.g., [27, 10, 11];
- (7) theory of operator algebras. See, e.g., [15], where the closedness of finite sums of full Fock spaces over subspaces of \mathbb{C}^d plays a crucial role for construction of a topological isomorphism between universal operator algebras;
- (8) quadratic programming. See, e.g., [28];

and others.

1.4. In this paper we study Questions 1 and 2. We provide a sufficient condition for the sum of complemented subspaces of a Banach space X to be complemented. Under this condition a formula for a projection onto the sum is given. We also show that the condition is sharp (in a certain sense). Special attention is paid to the case when X is a Hilbert space and P_1, \dots, P_n are orthogonal projections; in this case we get more precise results.

The paper is organized as follows. In Section 2 we study Questions 1 and 2 in the (general) Banach space setting. In Subsection 2.1 we make a few simple observations on the questions. In Subsection 2.2 we present known results. Our results are presented in Subsections 2.3, 2.4, and 2.5. Their proofs are given in Subsections 2.6 and 2.7.

In Section 3 we study Questions 1 and 2 in the Hilbert space setting. The structure of the section is similar to that of Section 2. Our results are presented in Subsections 3.3 and 3.4.

Finally, in the Appendix we present the Önal-Yurdakul proof of their result on the sum of two complemented subspaces (for more details see Subsection 2.2).

1.5. Notation. Throughout the paper, X is a real or complex Banach space with norm $\|\cdot\|$. When X is a Hilbert space, we denote by $\langle \cdot, \cdot \rangle$ the corresponding inner product. By a subspace we mean a linear set. The identity operator on X is denoted by I (throughout the paper it is clear which Banach space is being considered). All operators in the paper are continuous linear operators. In particular, by a projection we always mean a continuous linear projection. The kernel and range of an operator T will be denoted by $\ker(T)$ and $\text{Ran}(T)$, respectively. All vectors are vector-columns; the letter "t" means transpose.

2. THE BANACH SPACE CASE

Let X be a Banach space, X_1, \dots, X_n be complemented subspaces of X , and P_1, \dots, P_n be projections onto X_1, \dots, X_n , respectively.

2.1. Simple observations. We begin with a few simple observations on Questions 1 and 2. These observations were used by many authors.

- (1) If $P_i|_{X_j} = 0$ for all $i \neq j$, $i, j \in \{1, \dots, n\}$, then $X_1 + \dots + X_n$ is complemented in X and

$$P = P_1 + \dots + P_n$$

is a projection onto $X_1 + \dots + X_n$.

- (2) Let $n = 2$. If $P_2|_{X_1} = 0$, that is, $P_2P_1 = 0$, then $X_1 + X_2$ is complemented in X and

$$P = P_1 + P_2 - P_1P_2$$

is a projection onto $X_1 + X_2$. Note that $P_1 + P_2 - P_1P_2 = I - (I - P_1)(I - P_2)$. Now an induction argument shows that if $P_i|_{X_j} = 0$ for all $i > j$, $i, j \in \{1, \dots, n\}$, then $X_1 + \dots + X_n$ is complemented in X and

$$P = I - (I - P_1)(I - P_2)\dots(I - P_n)$$

is a projection onto $X_1 + \dots + X_n$.

- (3) (see, e.g., [30, Lemma 2.6]) Let $n = 2$. If X_2 is finite dimensional, then $X_1 + X_2$ is complemented in X .

Indeed, we can assume that $X_2 \cap X_1 = \{0\}$. Using the Hahn-Banach theorem one can easily construct a projection P_2 onto X_2 such that $P_2|_{X_1} = 0$. Now observation (2) shows that $X_1 + X_2$ is complemented in X .

2.2. Known results. Questions 1 and 2 seem to be very basic in the theory of complemented subspaces, but, to our knowledge, there are only a few known results (in the general Banach space setting). Let us present them.

For $n = 2$ each of the following conditions is sufficient for $X_1 + X_2$ to be complemented in X :

- (1) (Alan LaVergne, 1979, [19, Proposition]) P_2P_1 is strictly singular. In fact, the proof given in [19] works for the case when $I - P_2P_1$ is Fredholm of index zero;
- (2) (Lars Svensson, 1987, [30, Lemma 2.5]) $\ker(I - P_2P_1) = \ker(I - P_1P_2) = X_1 \cap X_2$ is complemented in X and $\text{Ran}(I - P_2P_1)$, $\text{Ran}(I - P_1P_2)$ are also complemented in X ;
- (3) ([30, Theorem 2.8]) $I - P_2P_1$ and $I - P_1P_2$ are Fredholm of index zero. In fact, the proof given in [30] works for the case when $I - P_2P_1$ and $I - P_1P_2$ are Fredholm;
- (4) (Manuel Gonzalez, 1994, [14, Lemma 1]) P_2P_1 is inessential. The proof given in [14] repeats that of [19] (note that if an operator $A : X \rightarrow X$ is inessential, then $I - A$ is Fredholm of index zero).
- (5) (Süleyman Önal and Murat Yurdakul, 2013, [20]) the restriction of the operator $I - P_2P_1$ to its invariant subspace X_2 is Fredholm. One can easily check that the condition is equivalent to the following: the operator $I - P_2P_1$ is Fredholm. Önal and Yurdakul gave a simple and elegant proof of the result. Unfortunately, the paper [20] seems to be unavailable in the Internet. Therefore, for convenience of the reader, we present the proof in the Appendix.

We should note that the proof shows more. Namely, if $\ker(I - P_2P_1)$ is finite dimensional and $(I - P_2P_1)(X_2)$ is complemented in X_2 , then $X_1 + X_2$ is complemented in X .

Concerning Question 2, a few formulas for a projection onto $X_1 + X_2$ (under certain conditions) can be found in [30]. For example, if $\ker(I - P_2P_1) = \ker(I - P_1P_2) = \{0\}$ and $\text{Ran}(I - P_2P_1)$, $\text{Ran}(I - P_1P_2)$ are complemented in X , then

$$P = P_1A_{21}(I - P_2) + P_2A_{12}(I - P_1)$$

is a projection onto $X_1 + X_2$, here A_{12} and A_{21} are left-inverses for $I - P_1P_2$ and $I - P_2P_1$, respectively. One more formula can be obtained by the proof of the Önal-Yurdakul result.

For an arbitrary n each of the following conditions is sufficient for $X_1 + \dots + X_n$ to be complemented in X :

- (1) ([19, Corollary]) X_1, \dots, X_n are pairwise totally incomparable. We should note that using LaVergne's proof of [19, Proposition] one can get a stronger result. In fact, using the proof one can easily show that if P_2P_1 is strictly singular, then there exists a projection P onto $X_1 + X_2$ such that P equals $P_1 + P_2 - P_1P_2$ modulo strictly singular operators. Now an induction argument shows that if P_iP_j is strictly singular for each pair $i > j$, $i, j \in \{1, \dots, n\}$, then $X_1 + \dots + X_n$ is complemented in X and there exists a projection P onto $X_1 + \dots + X_n$ such that P equals

$$I - (I - P_1)\dots(I - P_n)$$

modulo strictly singular operators.

- (2) ([30, Corollary 2.9]) $P_i P_j$ is compact for every pair $i \neq j$, $i, j \in \{1, \dots, n\}$. Moreover, under this condition there exists a projection P onto $X_1 + \dots + X_n$ such that P equals

$$P_1 + \dots + P_n$$

modulo compact operators.

2.3. Our result. In this subsection we provide a sufficient condition for $X_1 + \dots + X_n$ to be complemented in X . Under the condition a formula for a projection onto the sum is given. The result can be regarded as a strengthening of observation (1) in Subsection 2.1.

Suppose that nonnegative numbers ε_{ij} , $i \neq j$, $i, j \in \{1, \dots, n\}$ are such that

$$(2.1) \quad \|P_i x\| \leq \varepsilon_{ij} \|x\|, \quad x \in X_j$$

for every $i \neq j$, $i, j \in \{1, \dots, n\}$.

Remark 2.1. It is clear that (2.1) is equivalent to the inequality $\|P_i|_{X_j}\| \leq \varepsilon_{ij}$. The reader may wonder why we don't set $\varepsilon_{ij} := \|P_i|_{X_j}\|$. Answer: we believe that (2.1) is more convenient for applications. Indeed, finding the exact value of $\|P_i|_{X_j}\|$ is usually much more difficult than obtaining an inequality of the form (2.1).

Define the $n \times n$ matrix $E = (e_{ij})$ by

$$e_{ij} = \begin{cases} 0, & \text{if } i = j; \\ \varepsilon_{ij}, & \text{if } i \neq j. \end{cases}$$

Denote by $r(E)$ the spectral radius of E . Set $A := P_1 + \dots + P_n$.

Now we are ready to formulate our first result.

Theorem 2.1. *If $r(E) < 1$, then $X_1 + \dots + X_n$ is complemented in X and the subspace*

$$\{x \in X \mid P_1 x = 0, \dots, P_n x = 0\}$$

is a complement of $X_1 + \dots + X_n$ in X . Moreover, the sequence of operators

$$I - (I - A)^N$$

converges uniformly to a projection P onto $X_1 + \dots + X_n$ as $N \rightarrow \infty$.

Remark 2.2. For $n = 2$ the inequality $r(E) < 1$ is equivalent to

$$\varepsilon_{12}\varepsilon_{21} < 1.$$

For $n = 3$ the inequality $r(E) < 1$ is equivalent to

$$\varepsilon_{12}\varepsilon_{21} + \varepsilon_{23}\varepsilon_{32} + \varepsilon_{31}\varepsilon_{13} + \varepsilon_{12}\varepsilon_{23}\varepsilon_{31} + \varepsilon_{21}\varepsilon_{32}\varepsilon_{13} < 1.$$

For any $n \geq 2$, $r(E) < 1$ if and only if each principal minor of the matrix $I - E$ is positive. (Recall that a principal minor is the determinant of a principal submatrix; a principal submatrix is a square submatrix obtained by removing certain rows and columns with the same index sets.) This fact is an easy consequence of the theory of nonnegative matrices (see, e.g., [16, Chapter 8]).

2.4. A rate of convergence. For practical applications it is important to know how fast does the sequence $I - (I - A)^N$ converge to P . Our next result shows that the rate of convergence can be estimated from above by $C\alpha^N$, where $\alpha \in [0, 1)$. To formulate the result we need the following notation: for two vectors $u, v \in \mathbb{R}^n$ we will write $u \leq v$ if $u \leq v$ coordinatewise.

Theorem 2.2. *The following statements on the rate of convergence of $I - (I - A)^N$ to P are true.*

- (1) *Suppose a vector $w = (w_1, \dots, w_n)^t$ with positive coordinates and a number $\alpha \in [0, 1)$ satisfy $EW \leq \alpha w$. Then*

$$\|I - (I - A)^N - P\| \leq (w_1 + \dots + w_n) \max\{(1/w_1)\|P_1\|, \dots, (1/w_n)\|P_n\|\} \frac{\alpha^N}{1 - \alpha}$$

for any $N \geq 1$.

- (2) *Suppose a vector $w = (w_1, \dots, w_n)^t$ with positive coordinates and a number $\alpha \in [0, 1)$ satisfy $w^t E \leq \alpha w^t$. Then*

$$\|I - (I - A)^N - P\| \leq (w_1\|P_1\| + \dots + w_n\|P_n\|) \max\{(1/w_1), \dots, (1/w_n)\} \frac{\alpha^N}{1 - \alpha}$$

for any $N \geq 1$.

Remark 2.3. Since E is a nonnegative matrix, the existence of a vector $w \in \mathbb{R}^n$ with positive coordinates and a number $\alpha \in [0, 1)$ such that $EW \leq \alpha w$ is equivalent to $r(E) < 1$. More precisely, if such w and α exist, then $r(E) \leq \alpha < 1$ (see [16, Corollary 8.1.29]). Conversely, suppose that $r(E) < 1$. If E is irreducible, then one can take α to be $r(E)$ and w a Perron-Frobenius vector of E . If E is not irreducible, then consider the matrix $E' = (e_{ij} + \delta)$ for sufficiently small $\delta > 0$, and take α to be $r(E')$ and w a Perron-Frobenius vector of E' .

Similarly, the existence of a vector w with positive coordinates and a number $\alpha \in [0, 1)$ such that $w^t E \leq \alpha w^t$ is equivalent to $r(E) < 1$.

Using Theorem 2.2, we can get concrete estimates for the rate of convergence of $I - (I - A)^N$ to P . Suppose E is irreducible and $r(E) < 1$. Take α to be $r(E)$ and w a Perron-Frobenius vector of E . Then we get

$$\|I - (I - A)^N - P\| \leq (w_1 + \dots + w_n) \max\{(1/w_1)\|P_1\|, \dots, (1/w_n)\|P_n\|\} \frac{(r(E))^N}{1 - r(E)}.$$

Similarly, we can take α to be $r(E)$ and w a left Perron-Frobenius vector of E . Then we get

$$\|I - (I - A)^N - P\| \leq (w_1\|P_1\| + \dots + w_n\|P_n\|) \max\{(1/w_1), \dots, (1/w_n)\} \frac{(r(E))^N}{1 - r(E)}.$$

Remark 2.4. In the study of Questions 1 and 2 one can assume that E is irreducible. Indeed, suppose that E is reducible and $r(E) < 1$. Then, up to a permutation of the

subspaces X_1, \dots, X_n , the matrix E has the form

$$E = \begin{pmatrix} E_1 & * & \dots & * \\ 0 & E_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \dots & 0 & E_m \end{pmatrix},$$

where E_1, \dots, E_m are irreducible and $r(E_i) < 1$ for $i = 1, \dots, m$. Now we apply Theorem 2.1 to the first group of subspaces (i.e. X_1, \dots, X_{n_1} , where n_1 is order of the matrix E_1) with the corresponding matrix E_1 . Then we see that their sum \tilde{X}_1 is complemented in X and $I - (I - A_1)^N$ converges to a projection \tilde{P}_1 onto \tilde{X}_1 as $N \rightarrow \infty$. Similarly, we apply Theorem 2.1 to each of the remaining $m - 1$ groups of subspaces. Then we see that \tilde{X}_i , the sum of subspaces of the i -th group, is complemented in X and $I - (I - A_i)^N$ converges to a projection \tilde{P}_i onto \tilde{X}_i as $N \rightarrow \infty$, $i = 1, \dots, m$. Clearly, $\tilde{P}_i|_{\tilde{X}_j} = 0$ for every pair $i > j$. Now observation (2) in Subsection 2.1 shows that $\tilde{X}_1 + \dots + \tilde{X}_m = X_1 + \dots + X_n$ is complemented in X and

$$I - (I - \tilde{P}_1) \dots (I - \tilde{P}_m)$$

is a projection onto $X_1 + \dots + X_n$.

2.5. On the necessity of the condition $r(E) < 1$. The assumption $r(E) < 1$ is a *sharp* sufficient condition for $X_1 + \dots + X_n$ to be complemented in X . More precisely, we have the following result.

Theorem 2.3. *Let $E = (e_{ij})$ be an $n \times n$ matrix with $e_{ii} = 0$ for $i = 1, \dots, n$ and $e_{ij} \geq 0$ for any $i \neq j$. If $r(E) = 1$, then there exist a Banach space X , complemented subspaces X_1, \dots, X_n of X , and projections P_1, \dots, P_n onto X_1, \dots, X_n , respectively, such that*

- (1) $\|P_i x\| = e_{ij} \|x\|$, $x \in X_j$, for each pair $i \neq j$, $i, j \in \{1, \dots, n\}$;
- (2) $X_1 + \dots + X_n$ is not complemented in X .

2.6. Proof of Theorems 2.1 and 2.2. First, let us prove Theorem 2.1 and the first part of Theorem 2.2. Thus we assume that a vector $w = (w_1, \dots, w_n)^t$ with positive coordinates and a number $\alpha \in [0, 1)$ satisfy $Ew \leq \alpha w$.

Let $X_1 \times \dots \times X_n$ be the linear space of all vector-columns $(x_1, \dots, x_n)^t$ with $x_1 \in X_1, \dots, x_n \in X_n$ endowed with the weighted ∞ -norm

$$\|(x_1, \dots, x_n)^t\| = \max\{(1/w_1)\|x_1\|, \dots, (1/w_n)\|x_n\|\}.$$

Then, obviously, $X_1 \times \dots \times X_n$ is a Banach space. Define the operator $S : X_1 \times \dots \times X_n \rightarrow X$ by

$$S(x_1, \dots, x_n)^t = x_1 + \dots + x_n, \quad x_1 \in X_1, \dots, x_n \in X_n,$$

and the operator $J : X \rightarrow X_1 \times \dots \times X_n$ by

$$Jx = (P_1 x, \dots, P_n x)^t, \quad x \in X.$$

Then $SJ = P_1 + \dots + P_n = A$. Set

$$G = JS : X_1 \times \dots \times X_n \rightarrow X_1 \times \dots \times X_n.$$

Let $(G_{ij} : X_j \rightarrow X_i \mid i, j = 1, \dots, n)$ be the block decomposition of G . It is clear that G_{ij} acts as P_i on X_j . In particular, $G_{ii} = I$ for $i = 1, \dots, n$.

Let us show that G is invertible. To this end we will estimate $\|G - I\|$. For the block decomposition of $G - I$ we have $(G - I)_{ii} = 0$ for $i = 1, \dots, n$, and $(G - I)_{ij} = G_{ij}$ for $i \neq j$. Then $\|(G - I)_{ij}\| \leq \varepsilon_{ij}$ for $i \neq j$ and thus $\|(G - I)_{ij}\| \leq e_{ij}$ for every pair i, j . It follows easily that $\|G - I\| \leq \|E\|$, where $\|E\|$ is the operator norm of the matrix E considered as the operator on the space \mathbb{R}^n endowed with the weighted ∞ -norm $\|u\| = \max\{(1/w_1)|u_1|, \dots, (1/w_n)|u_n|\}$, $u = (u_1, \dots, u_n)^t \in \mathbb{R}^n$. But

$$\|E\| = \max\{(e_{i1}w_1 + e_{i2}w_2 + \dots + e_{in}w_n)/w_i \mid i = 1, \dots, n\} \leq \alpha.$$

Therefore $\|G - I\| \leq \alpha < 1$. Consequently, G is invertible and

$$(2.2) \quad G^{-1} = (I - (I - G))^{-1} = \sum_{k=0}^{\infty} (I - G)^k,$$

where the series converges uniformly.

Now we claim that $P = SG^{-1}J : X \rightarrow X$ is a projection onto $X_1 + \dots + X_n$. Indeed, the operator P has the following properties:

- (1) P is a continuous linear operator;
- (2) $\text{Ran}(P) \subset \text{Ran}(S) = X_1 + \dots + X_n$;
- (3) for any $x \in X_1 + \dots + X_n$ $Px = x$. Indeed, $x = Sv$ for some $v \in X_1 \times \dots \times X_n$. Then

$$Px = SG^{-1}JSv = SG^{-1}Gv = Sv = x.$$

These three properties of P imply that P is a projection onto $X_1 + \dots + X_n$. Hence $X_1 + \dots + X_n$ is complemented in X .

Further, $\ker(P)$ is a complement of $X_1 + \dots + X_n$ in X . It is easily seen that

$$\ker(P) = \ker(J) = \{x \in X \mid P_1x = 0, \dots, P_nx = 0\}.$$

Let us show that the sequence of operators $I - (I - A)^N$ converges uniformly to P as $N \rightarrow \infty$. Using (2.2) we get

$$P = S \left(\sum_{k=0}^{\infty} (I - G)^k \right) J = \lim_{N \rightarrow \infty} S \left(\sum_{k=0}^{N-1} (I - G)^k \right) J.$$

Since

$$S(I - G) = S(I - JS) = (I - SJ)S = (I - A)S$$

we see that $S(I - G)^k = (I - A)^k S$ for any $k = 0, 1, \dots$. Therefore

$$\begin{aligned} P &= \lim_{N \rightarrow \infty} \left(\sum_{k=0}^{N-1} (I - A)^k S \right) J = \lim_{N \rightarrow \infty} \left(\sum_{k=0}^{N-1} (I - A)^k \right) A = \\ &= \lim_{N \rightarrow \infty} \left(\sum_{k=0}^{N-1} (I - A)^k \right) (I - (I - A)) = \lim_{N \rightarrow \infty} (I - (I - A)^N). \end{aligned}$$

This finishes the proof of Theorem 2.1.

It remains to estimate $\|I - (I - A)^N - P\|$. We have

$$\|I - (I - A)^N - P\| = \|S \left(\sum_{k=N}^{\infty} (I - G)^k \right) J\| \leq \|S\| \|J\| \sum_{k=N}^{\infty} \|I - G\|^k.$$

From the definitions of S and J we have

$$\|S\| \leq w_1 + \dots + w_n$$

and

$$\|J\| = \max\{(1/w_1)\|P_1\|, \dots, (1/w_n)\|P_n\|\}.$$

Also, recall that $\|G - I\| \leq \alpha$. Therefore

$$\|I - (I - A)^N - P\| \leq (w_1 + \dots + w_n) \max\{(1/w_1)\|P_1\|, \dots, (1/w_n)\|P_n\|\} \frac{\alpha^N}{1 - \alpha}.$$

This finishes the proof of the first part of Theorem 2.2.

The proof of the second part of Theorem 2.2 follows the same lines as the one for the first part but with the only difference: instead of the weighted ∞ -norm on the linear space $X_1 \times \dots \times X_n$ one should consider the weighted 1-norm

$$\|(x_1, \dots, x_n)^t\| = w_1\|x_1\| + \dots + w_n\|x_n\|.$$

2.7. Proof of Theorem 2.3. Our construction of a space X , its subspaces X_1, \dots, X_n , and projections P_1, \dots, P_n is based on the following simple observation. Let $\langle \cdot, \cdot \rangle$ be the standard inner product in \mathbb{R}^n , i.e.,

$$\langle u, v \rangle = u_1 v_1 + \dots + u_n v_n,$$

where $u = (u_1, \dots, u_n)^t$ and $v = (v_1, \dots, v_n)^t$. Each nonzero vector $v \in \mathbb{R}^n$ spans the one-dimensional subspace

$$L_{\mathbb{R}}(v) = \{\lambda v \mid \lambda \in \mathbb{R}\} = \{(\lambda v_1, \dots, \lambda v_n)^t \mid \lambda \in \mathbb{R}\}.$$

If a vector $u \in \mathbb{R}^n$ satisfies $\langle v, u \rangle = 1$, then the mapping

$$x \mapsto \langle x, u \rangle v, \quad x \in \mathbb{R}^n$$

is a projection onto $L_{\mathbb{R}}(v)$.

To construct a space X , its subspaces X_1, \dots, X_n , and projections P_1, \dots, P_n we need two collections of vectors $u^{(i)} \in \mathbb{R}^n$, $i = 1, \dots, n$ and $v^{(j)} \in \mathbb{R}^n$, $j = 1, \dots, n$ which have the following properties:

- (1) $v^{(1)}, \dots, v^{(n)}$ are unit basis vectors of \mathbb{R}^n ;
- (2) $\langle v^{(i)}, u^{(i)} \rangle = 1$ for $i = 1, \dots, n$ and $|\langle v^{(j)}, u^{(i)} \rangle| = e_{ij}$ for each pair $i \neq j$, $i, j \in \{1, \dots, n\}$;
- (3) the n -th coordinate of the vectors $u^{(1)}, \dots, u^{(n)}$ equals 0.

Such vectors can be constructed as follows. Let $f^{(i)} \in \mathbb{R}^n$ be the transpose of the i -th row of the matrix $I - E$, i.e.,

$$f^{(i)} = (-e_{i1}, \dots, -e_{i,i-1}, 1, -e_{i,i+1}, \dots, -e_{i,n})^t, \quad i = 1, \dots, n.$$

Denote by $g^{(j)}$, $j = 1, \dots, n$, the standard unit basis vectors of \mathbb{R}^n , i.e.,

$$g^{(j)} = (0, \dots, 0, 1, 0, \dots, 0)^t, \quad j = 1, \dots, n,$$

where the 1 is in the j -th position. Clearly, $\langle g^{(i)}, f^{(i)} \rangle = 1$ for $i = 1, \dots, n$ and $\langle g^{(j)}, f^{(i)} \rangle = -e_{ij}$ for each pair $i \neq j$, $i, j \in \{1, \dots, n\}$. Further, since E is a nonnegative matrix, we conclude that $r(E) = 1$ is an eigenvalue of E . This means that the matrix $I - E$ is singular, i.e., the vectors $f^{(1)}, \dots, f^{(n)}$ are linearly dependent. It follows that the dimension of the linear span of $f^{(1)}, \dots, f^{(n)}$ is not greater than $n - 1$. Thus there exists a unitary operator $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that $T(\text{linear span of } f^{(1)}, \dots, f^{(n)})$ is contained in the hyperplane $\{u = (u_1, \dots, u_n)^t \in \mathbb{R}^n \mid u_n = 0\}$. Set $u^{(i)} = Tf^{(i)}$, $i = 1, \dots, n$ and $v^{(j)} = Tg^{(j)}$, $j = 1, \dots, n$. It is clear that these two collections of vectors have the required properties.

Now we are ready to construct a space X , its subspaces X_1, \dots, X_n , and projections P_1, \dots, P_n . Let Y be a closed uncomplemented subspace of a Banach space Z . Define X to be the linear space

$$\underbrace{Y \times \dots \times Y}_{n-1} \times Z$$

of all vector-columns $x = (y_1, \dots, y_{n-1}, z)^t$ with $y_1 \in Y, \dots, y_{n-1} \in Y, z \in Z$ endowed with the norm

$$\|x\| = (\|y_1\|^2 + \dots + \|y_{n-1}\|^2 + \|z\|^2)^{1/2}.$$

Then, obviously, X is a Banach space.

To make our construction of subspaces X_1, \dots, X_n and projections P_1, \dots, P_n more clear we introduce the following notation. For $y \in Y$ and $v = (v_1, \dots, v_n)^t \in \mathbb{R}^n$ we set

$$yv := (v_1 y, \dots, v_n y)^t.$$

For $x = (y_1, \dots, y_{n-1}, z) \in X$ and $u = (u_1, \dots, u_n)^t \in \mathbb{R}^n$ set

$$\langle x, u \rangle = u_1 y_1 + \dots + u_{n-1} y_{n-1} + u_n z.$$

Now for each $i = 1, \dots, n$ we define the subspace X_i of X by

$$X_i = L_Y(v^{(i)}) = \{yv^{(i)} \mid y \in Y\} = \{(v_1^{(i)} y, \dots, v_n^{(i)} y)^t \mid y \in Y\}$$

and the projection $P_i : X \rightarrow X$ onto X_i by

$$\begin{aligned} P_i x &= \langle x, u^{(i)} \rangle v^{(i)} = \\ &= (v_1^{(i)}(u_1^{(i)} y_1 + \dots + u_{n-1}^{(i)} y_{n-1} + u_n^{(i)} z), \dots, v_n^{(i)}(u_1^{(i)} y_1 + \dots + u_{n-1}^{(i)} y_{n-1} + u_n^{(i)} z)) = \\ &= (v_1^{(i)}(u_1^{(i)} y_1 + \dots + u_{n-1}^{(i)} y_{n-1}), \dots, v_n^{(i)}(u_1^{(i)} y_1 + \dots + u_{n-1}^{(i)} y_{n-1})). \end{aligned}$$

Let us show that $\|P_i x\| = e_{ij} \|x\|$, $x \in X_j$, for each pair $i \neq j$, $i, j \in \{1, \dots, n\}$. Consider arbitrary $x \in X_j$. Then $x = yv^{(j)}$ for some $y \in Y$. Since $v^{(j)}$ is a unit vector, we see that $\|x\| = \|y\|$. We have

$$P_i x = \langle yv^{(j)}, u^{(i)} \rangle v^{(i)} = (\langle v^{(j)}, u^{(i)} \rangle y) v^{(i)}.$$

Therefore

$$\|P_i x\| = \|\langle v^{(j)}, u^{(i)} \rangle y\| = |\langle v^{(j)}, u^{(i)} \rangle| \|y\| = e_{ij} \|x\|.$$

It remains to show that $X_1 + \dots + X_n$ is not complemented in X . Since $v^{(1)}, \dots, v^{(n)}$ are linearly independent, we conclude that $X_1 + \dots + X_n = Y \times \dots \times Y$. Recall that Y is not complemented in Z ; it follows that $X_1 + \dots + X_n$ is not complemented in $X = Y \times \dots \times Y \times Z$.

3. HILBERT SPACE CASE

Now we consider the case when X is a Hilbert space. In this case a subspace is complemented if and only if it is closed, and hence Question 1 is equivalent to Question 3. Concerning Question 2, it is natural to consider *orthogonal* projections.

So, let X be a Hilbert space, X_1, \dots, X_n be closed subspaces of X , and P_1, \dots, P_n be the orthogonal projections onto X_1, \dots, X_n , respectively.

3.1. Simple observation. We begin with the following simple observation which was used by many authors: if $P_i|_{X_j} = 0$, that is, X_j is orthogonal to X_i , for every pair $i \neq j$, $i, j \in \{1, \dots, n\}$, then $X_1 + \dots + X_n$ is closed and

$$P = P_1 + \dots + P_n$$

is the orthogonal projection onto $X_1 + \dots + X_n$.

3.2. Known results. We begin with $n = 2$. There are many criteria for the sum of two closed subspaces to be closed, see, e.g., [9, Lemma 11, Theorems 12, 13], [3, Proposition 5.16], [8, Corollary 2.12], [12] and references therein. We should note that much of them belong to the mathematical folklore.

Now consider Question 2. First, we mention the following well-known result: if $X_1 \cap X_2 = \{0\}$ and $X_1 + X_2$ is closed, then

$$P = P_1(I - P_2P_1)^{-1}(I - P_2) + P_2(I - P_1P_2)^{-1}(I - P_1)$$

is the orthogonal projection onto $X_1 + X_2$. Some other formulas can be obtained by the sum-intersection duality. Since $(X_1 + X_2)^\perp = X_1^\perp \cap X_2^\perp$, we see that $\overline{X_1 + X_2} = (X_1^\perp \cap X_2^\perp)^\perp$. If $X_1 + X_2$ is closed, then we get $X_1 + X_2 = (X_1^\perp \cap X_2^\perp)^\perp$. Now using formulas for the orthogonal projection onto the intersection of two closed subspaces one can easily get formulas for the orthogonal projection onto the closed sum of two closed subspaces. We present some examples. Let Y_1, Y_2 be closed subspaces of X , and Q_1, Q_2 be the orthogonal projections onto Y_1, Y_2 , respectively.

- (1) By the famous result of John von Neumann the sequence $(Q_2Q_1)^N$ converges strongly to the projection Q onto $Y_1 \cap Y_2$ as $N \rightarrow \infty$. Moreover, if $Y_1 + Y_2$ is closed, then $(Q_2Q_1)^N$ converges to Q uniformly (see, e.g., [9]). Conclusion: if $X_1 + X_2$ is closed, then the sequence

$$I - ((I - P_2)(I - P_1))^N$$

converges uniformly to the orthogonal projection P onto $X_1 + X_2$ (note that the closedness of $X_1 + X_2$ implies the closedness of $X_1^\perp + X_2^\perp$).

- (2) If $Y_1 + Y_2$ is closed, then $Q = 2(Q_1 : Q_2)$ is the orthogonal projection onto $Y_1 \cap Y_2$ (see [1]). Here " $:$ " means the parallel sum of two operators. Conclusion: if $X_1 + X_2$ is closed, then

$$P = I - 2((I - P_1) : (I - P_2))$$

is the orthogonal projection onto $X_1 + X_2$.

Some other formulas for the orthogonal projections onto the sum and intersection of subspaces in the case where X is finite-dimensional can be found in [24].

Now we turn to the case when n is arbitrary. Criteria for the sum of n closed subspaces to be closed can be found in [30], [3, Theorem 5.19], [12], [2, Theorem 4.1], [15], [13].

Let us consider Question 2. As in the case $n = 2$, some formulas for the orthogonal projection onto the closed sum of n closed subspaces can be obtained by the sum-intersection duality. Since $(X_1 + \dots + X_n)^\perp = X_1^\perp \cap \dots \cap X_n^\perp$, we see that $\overline{X_1 + \dots + X_n} = (X_1^\perp \cap \dots \cap X_n^\perp)^\perp$. If $X_1 + \dots + X_n$ is closed, then we get $X_1 + \dots + X_n = (X_1^\perp \cap \dots \cap X_n^\perp)^\perp$. Now using formulas for the orthogonal projection onto the intersection of n closed subspaces one can easily get formulas for the orthogonal projection onto the closed sum of n closed subspaces.

Many formulas for the orthogonal projection onto the intersection of n subspaces can be obtained by the *method of random alternating projections* and its modifications. As an example, we consider the *method of cyclic alternating projections*. Let Y_1, \dots, Y_n be closed subspaces of X and Q_1, \dots, Q_n be the orthogonal projections onto Y_1, \dots, Y_n , respectively. By the famous result of Israel Halperin the sequence $(Q_n \dots Q_1)^N$ converges strongly to the orthogonal projection Q onto $Y_1 \cap \dots \cap Y_n$ as $N \rightarrow \infty$. Moreover, if $Y_1^\perp + \dots + Y_n^\perp$ is closed, then $(Q_n \dots Q_1)^N$ converges to Q uniformly (see, e.g., [9, Section 3] and [4, Section 3.7], or [2, Section 4]). Conclusion: if $X_1 + \dots + X_n$ is closed, then the sequence

$$I - ((I - P_n) \dots (I - P_1))^N$$

converges uniformly to the orthogonal projection P onto $X_1 + \dots + X_n$.

Some other formulas for the orthogonal projection onto the intersection of n subspaces can be found in [26], [5].

3.3. Our result. In this subsection we provide a sufficient condition for $X_1 + \dots + X_n$ to be closed in X . Under the condition a formula for the orthogonal projection onto the sum is given. The result can be regarded as a strengthening of the simple observation in Subsection 3.1.

As in the Banach space setting, we assume that nonnegative numbers ε_{ij} , $i \neq j$, $i, j \in \{1, \dots, n\}$ are such that

$$(3.1) \quad \|P_i x\| \leq \varepsilon_{ij} \|x\|, \quad x \in X_j$$

for every $i \neq j$, $i, j \in \{1, \dots, n\}$. One can easily check that (3.1) is equivalent to the inequality $\|P_i P_j\| \leq \varepsilon_{ij}$. Now observe that $\|P_i P_j\| = \|(P_j P_i)^*\| = \|P_j P_i\|$. Hence, it is natural to assume that $\varepsilon_{ij} = \varepsilon_{ji}$ for every pair $i \neq j$, $i, j \in \{1, \dots, n\}$. Define the $n \times n$ matrix $E = (e_{ij})$ by

$$e_{ij} = \begin{cases} 0, & \text{if } i = j; \\ \varepsilon_{ij}, & \text{if } i \neq j. \end{cases}$$

It is clear that E is symmetric and nonnegative. It follows that $r(E)$, the spectral radius of E , is the maximum eigenvalue of E . Set $A := P_1 + \dots + P_n$.

Now we are ready to formulate our result.

Theorem 3.1. *If $r(E) < 1$, then $X_1 + \dots + X_n$ is closed in X . Moreover, the sequence of operators*

$$I - (I - A)^N$$

converges uniformly to the orthogonal projection P onto $X_1 + \dots + X_n$ as $N \rightarrow \infty$. For the rate of convergence we have

$$\|I - (I - A)^N - P\| \leq (r(E))^N$$

for any $N \geq 1$.

Theorem 3.1 is the Hilbert space analogue of Theorems 2.1 and 2.2. It is noteworthy that in the Hilbert space setting we have more precise estimate for the rate of convergence than in the Banach space setting.

3.4. On the necessity of the condition $r(E) < 1$. The assumption $r(E) < 1$ is a *sharp* sufficient condition for $X_1 + \dots + X_n$ to be closed in X . More precisely, we will prove the following result, which is the Hilbert space analogue of Theorem 2.3.

Theorem 3.2. *Let $E = (e_{ij})$ be a symmetric $n \times n$ matrix with $e_{ii} = 0$ for $i = 1, \dots, n$ and $e_{ij} \geq 0$ for any $i \neq j$. If $r(E) = 1$, then there exist a Hilbert space X and closed subspaces X_1, \dots, X_n of X such that*

- (1) $\|P_i|_{X_j}\| = e_{ij}$ for any $i \neq j$, $i, j \in \{1, \dots, n\}$; here P_i is the orthogonal projection onto X_i , $i = 1, \dots, n$.
- (2) $X_1 + \dots + X_n$ is not closed in X .

3.5. Proof of Theorem 3.1. To prove Theorem 3.1 we need the following lemma.

Lemma 3.1. *Let H, K be Hilbert spaces and $T : H \rightarrow K$ be a continuous linear operator. Suppose that*

$$\alpha\|x\| \leq \|Tx\| \leq \beta\|x\|, \quad x \in H,$$

where α, β are positive numbers. Then

$$\alpha\|y\| \leq \|T^*y\| \leq \beta\|y\|, \quad y \in \text{Ran}(T).$$

The proof is simple and is omitted.

Now we are ready to prove Theorem 3.1. For simplicity of notation, we set $r := r(E)$.

Let $X_1 \oplus \dots \oplus X_n$ be the (orthogonal) direct sum of Hilbert spaces X_1, \dots, X_n . Define the operator $S : X_1 \oplus \dots \oplus X_n \rightarrow X$ by

$$S(x_1, \dots, x_n)^t = x_1 + \dots + x_n, \quad x_1 \in X_1, \dots, x_n \in X_n,$$

and the operator $J : X \rightarrow X_1 \oplus \dots \oplus X_n$ by

$$Jx = (P_1x, \dots, P_nx)^t, \quad x \in X.$$

It is easily seen that $S^* = J$. We also have $SJ = P_1 + \dots + P_n = A$. Set

$$G = JS : X_1 \oplus \dots \oplus X_n \rightarrow X_1 \oplus \dots \oplus X_n.$$

Let $(G_{ij} : X_j \rightarrow X_i \mid i, j = 1, \dots, n)$ be the block decomposition of G . It is clear that G_{ij} acts as P_i on X_j . In particular, $G_{ii} = I$ for $i = 1, \dots, n$.

Let us estimate $\|G - I\|$. For the block decomposition of $G - I$ we have $(G - I)_{ii} = 0$ for $i = 1, \dots, n$, and $(G - I)_{ij} = G_{ij}$ for $i \neq j$. Then $\|(G - I)_{ij}\| \leq \varepsilon_{ij}$ for $i \neq j$ and thus $\|(G - I)_{ij}\| \leq e_{ij}$ for arbitrary $i, j \in \{1, \dots, n\}$. It follows easily that $\|G - I\| \leq \|E\|$, where $\|E\|$ is the operator norm of the matrix E considered as the operator on the space \mathbb{R}^n endowed with the standard inner product. Since E is a symmetric matrix, we conclude that $\|E\| = r$. Therefore $\|G - I\| \leq r$.

We claim that $X_1 + \dots + X_n$ is closed in X . To prove this, note that the inequality $r < 1$ implies that $\|G - I\| < 1$. It follows that G is invertible and consequently S is an isomorphic embedding. Then $\text{Ran}(S) = X_1 + \dots + X_n$ is closed in X .

Let us estimate $\|I - (I - A)^N - P\|$, where P is the orthogonal projection onto $X_1 + \dots + X_n$. Since $\|G - I\| \leq r$, we see that

$$|\langle (G - I)v, v \rangle| \leq r\|v\|^2, \quad v \in X_1 \oplus \dots \oplus X_n.$$

We have

$$\langle (G - I)v, v \rangle = \langle Gv, v \rangle - \|v\|^2 = \langle S^*Sv, v \rangle - \|v\|^2 = \|Sv\|^2 - \|v\|^2.$$

Hence,

$$(1 - r)\|v\|^2 \leq \|Sv\|^2 \leq (1 + r)\|v\|^2, \quad v \in X_1 \oplus \dots \oplus X_n.$$

Now Lemma 3.1 implies that

$$(1 - r)\|x\|^2 \leq \|S^*x\|^2 \leq (1 + r)\|x\|^2, \quad x \in \text{Ran}(S).$$

We have

$$\|S^*x\|^2 = \langle S^*x, S^*x \rangle = \langle SS^*x, x \rangle = \langle Ax, x \rangle.$$

Further, the closed subspace $\text{Ran}(S) = X_1 + \dots + X_n$ is invariant with respect to A . Denote by A' the restriction of A to $X_1 + \dots + X_n$. Then we get

$$(1 - r)\|x\|^2 \leq \langle A'x, x \rangle \leq (1 + r)\|x\|^2, \quad x \in X_1 + \dots + X_n,$$

and hence

$$|\langle (A' - I)x, x \rangle| \leq r\|x\|^2, \quad x \in X_1 + \dots + X_n.$$

Since $A' - I$ is self-adjoint, we conclude that $\|A' - I\| \leq r$.

Now we are ready to estimate $\|I - (I - A)^N - P\|$. Consider the orthogonal decomposition

$$X = (X_1 + \dots + X_n) \oplus (X_1 + \dots + X_n)^\perp = (X_1 + \dots + X_n) \oplus (X_1^\perp \cap \dots \cap X_n^\perp).$$

With respect to the decomposition we have

$$P = I \oplus 0, \quad A = A' \oplus 0.$$

Thus

$$I - (I - A)^N - P = -(I - A')^N \oplus 0$$

and

$$\|I - (I - A)^N - P\| = \|(I - A')^N\| \leq \|A' - I\|^N \leq r^N \rightarrow 0$$

as $N \rightarrow \infty$.

Theorem 3.1 is proved.

3.6. Proof of Theorem 3.2. For a number $\alpha \in (0, 1)$ consider the matrix $I - \alpha E$. Note that this matrix has the following properties:

- (1) $I - \alpha E$ is a real symmetric matrix with diagonal elements equal to 1;
- (2) the least eigenvalue of this matrix is equal to $1 - \alpha$. Consequently, this matrix is positive definite.

Therefore $I - \alpha E$ is the Gram matrix of some linearly independent collection of unit vectors of \mathbb{R}^n , say $v^{(i)} = v^{(i)}(\alpha)$, $i = 1, \dots, n$. Let $L_i = L_i(\alpha)$ be the one-dimensional subspace spanned by $v^{(i)}$, $i = 1, \dots, n$. Denote by $P_i = P_i(\alpha)$ the orthogonal projection onto L_i , $i = 1, \dots, n$. Clearly, $P_i v = \langle v, v^{(i)} \rangle v^{(i)}$, $v \in \mathbb{R}^n$, $i = 1, \dots, n$. It follows that

$$P_i v^{(j)} = \langle v^{(j)}, v^{(i)} \rangle v^{(i)} = -\alpha e_{ij} v^{(i)},$$

and consequently

$$\|P_i|_{L_j}\| = \alpha e_{ij}$$

for any $i \neq j$. Further, since 1 is an eigenvalue of E , we see that there exists a unit vector $c = (c_1, \dots, c_n) \in \mathbb{R}^n$ such that $Ec = c$. Then $(I - \alpha E)c = (1 - \alpha)c$ and consequently $\langle (I - \alpha E)c, c \rangle = 1 - \alpha$. We rewrite this equality as

$$\sum_{i,j} \langle v^{(j)}, v^{(i)} \rangle c_j c_i = 1 - \alpha,$$

which is equivalent to

$$(3.2) \quad \|c_1 v^{(1)} + \dots + c_n v^{(n)}\|^2 = 1 - \alpha.$$

Now we are ready to construct a Hilbert space X and its closed subspaces X_1, \dots, X_n with the needed properties. Take an arbitrary sequence $\alpha_k \in (0, 1)$, $k = 1, 2, \dots$, which converges to 1 as $k \rightarrow \infty$. Set

$$X = \bigoplus_{k=1}^{\infty} \mathbb{R}^n$$

and

$$X_i = \bigoplus_{k=1}^{\infty} L_i(\alpha_k), \quad i = 1, \dots, n,$$

where \bigoplus is the (orthogonal) direct sum of Hilbert spaces.

First, let us show that $\|P_i|_{X_j}\| = e_{ij}$ for any $i \neq j$ (here P_i is the orthogonal projection onto X_i). It is clear that

$$P_i = \bigoplus_{k=1}^{\infty} P_i(\alpha_k).$$

Hence,

$$\|P_i|_{X_j}\| = \sup\{\|P_i(\alpha_k)|_{L_j(\alpha_k)}\| \mid k = 1, 2, \dots\} = \sup\{\alpha_k e_{ij} \mid k = 1, 2, \dots\} = e_{ij}.$$

Let us show that $X_1 + \dots + X_n$ is not closed in X . Indeed, suppose that $X_1 + \dots + X_n$ is closed in X . Let $X_1 \oplus \dots \oplus X_n$ be the (orthogonal) direct sum of the Hilbert spaces X_1, \dots, X_n . Define the operator $S : X_1 \oplus \dots \oplus X_n \rightarrow X$ by

$$S(x_1, \dots, x_n)^t = x_1 + \dots + x_n, \quad x_1 \in X_1, \dots, x_n \in X_n.$$

This operator has the following properties:

- (1) $\ker(S) = \{0\}$. Indeed, suppose that $x_1 + \dots + x_n = 0$, where $x_1 \in X_1, \dots, x_n \in X_n$. Since the vectors $v^{(1)}(\alpha_k), \dots, v^{(n)}(\alpha_k)$ are linearly independent for $k = 1, 2, \dots$, we see that $x_1 = 0, \dots, x_n = 0$.
- (2) $\text{Ran}(S) = X_1 + \dots + X_n$ is closed in X .

These properties show that S is an isomorphic embedding. Hence, there exists a number $\beta > 0$ such that

$$\|Su\| \geq \beta\|u\|, \quad u \in X_1 \oplus \dots \oplus X_n.$$

We rewrite this inequality as

$$(3.3) \quad \|x_1 + \dots + x_n\|^2 \geq \beta^2(\|x_1\|^2 + \dots + \|x_n\|^2), \quad x_1 \in X_1, \dots, x_n \in X_n.$$

Now we choose

$$x_i = (0, \dots, 0, c_i v^{(i)}(\alpha_k), 0, \dots)^t, \quad i = 1, \dots, n.$$

By (3.3) and (3.2) we get $1 - \alpha_k \geq \beta^2$. But $\alpha_k \rightarrow 1$ as $k \rightarrow \infty$ and thus we get a contradiction. Hence, $X_1 + \dots + X_n$ is not closed in X .

Theorem 3.2 is proved.

4. APPENDIX: PROOF OF THE ÖNAL-YURDAKUL RESULT

We will present the Önal-Yurdakul proof of their result on the sum of two complemented subspaces. The result states that if the restriction of the operator $I - P_2 P_1$ to its invariant subspace X_2 is Fredholm, then $X_1 + X_2$ is complemented in X . The proof is based on the following elegant lemma.

Lemma 4.1. *Suppose that X'_2 is a subspace of X_2 . If the operator $I - P_2 P_1$ maps X'_2 isomorphically onto a complemented (in X_2) subspace of X_2 , then $X_1 + X'_2$ is complemented in X .*

Proof. Set $X''_2 := (I - P_2 P_1)(X'_2)$. By our assumption X''_2 is complemented in X_2 . Let $P''_2 : X_2 \rightarrow X_2$ be a projection onto X''_2 . Define the operator $A : X'_2 \rightarrow X''_2$ by $Ax = (I - P_2 P_1)x$, $x \in X'_2$. By our assumption A is an isomorphism.

Now we define the operator $P'_2 : X \rightarrow X$ by

$$P'_2 x := A^{-1} P''_2 P_2 (I - P_1)x, \quad x \in X.$$

It is clear that the operator P'_2 has the following properties:

- (1) P'_2 is a continuous linear operator;
- (2) $\text{Ran}(P'_2) \subset X'_2$;

(3) $P'_2 x = x$ for every $x \in X'_2$.

Therefore P'_2 is a projection onto X'_2 . Moreover, it is clear that $P'_2 x = 0$ for every $x \in X_1$. Now observation (2) in Subsection 2.1 implies that $X_1 + X'_2$ is complemented in X . \square

The Önal-Yurdakul result is a simple consequence of the lemma. Indeed, denote by T the restriction of the operator $I - P_2 P_1$ to its invariant subspace X_2 . Since T is Fredholm, we conclude that $\ker(T)$ is finite dimensional and hence complemented in X_2 . Let X'_2 be a complement of $\ker(T)$ in X_2 . Then $I - P_2 P_1$ maps X'_2 isomorphically onto $\text{Ran}(T)$ which is complemented in X_2 . By Lemma 4.1, $X_1 + X'_2$ is complemented in X . Now observation (3) in Subsection 2.1 implies that $X_1 + X_2 = (X_1 + X'_2) + \ker(T)$ is also complemented in X .

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